Discrete Mathematics 312 (2012) 3030–3036

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

On the intricacy of avoiding multiple-entry arrays

Lars-Daniel Öhman

Department of Mathematics and Mathematical Statistics, Umeå University, Sweden

ARTICLE INFO

Article history:
Received 8 July 2011
Received in revised form 13 June 2012
Accepted 2 July 2012
Available online 24 July 2012

Keywords:
Latin square
Constraint satisfaction
List-coloring

ABSTRACT

Let A be any $n \times n$ array on the symbols $[n] = \{1, \ldots, n\}$, with at most m symbols in each cell. An $n \times n$ Latin square L on the symbols $[n]$ is said to *avoid* A if no entry in L is present in the corresponding cell of A, and A is said to be *avoidable* if such a Latin square L exists. The *intricacy* of this problem is defined to be the minimum number of arrays into which A must be split in order to ensure that each part is avoidable. We present lower and upper bounds for the intricacy, and conjecture that the lower bound is in fact the correct answer.

1. Introduction

The concept of *intricacy* was first introduced by Daykin and Häggkvist in [6] for the problem of completing partial Latin squares, and a sample of the applications to this and other problems can be found in [12].

For the problem of completing partial Latin squares, the intricacy is the minimum $k \in \mathbb{N}$ such that any partial Latin square can be partitioned into at most k partial Latin squares, each of which is completable. In [12], the following theorem was proven.

Theorem 1.1 (*Opencomb*). *The intricacy of completing a partial Latin square is less than or equal to* 4.

For the general concept of intricacy, given a class of combinatorial construction problems, we ask first which problems are solvable. If all problem instances are solvable, the problem is dubbed *simple*, and we say that the intricacy is 1. An example of this situation is provided by the problem of finding a spanning tree in a graph, given a fixed partial spanning tree. If at least all minimal (with respect to some appropriate measure) problem instances are solvable, we say that the problem at hand is *fair*. For unfair problems we do not define the intricacy.

The intricacy for a general class of combinatorial construction problems is the minimum $k \in \mathbb{N}$ such that any problem instance can be partitioned into k or fewer solvable problem instances. In this paper, we will be concerned with the problem of producing a Latin square, under certain restrictions.

Definition 1.2. An $n \times n$ array A on the symbols $[n] = \{1, 2, 3, \ldots, n\}$ is *avoidable* if there exists an $n \times n$ Latin square L on the symbols $[n]$ such that no entry in L is present in the corresponding cell of A, and we say that such an L *avoids* A.

All arrays and Latin squares will henceforth be of order n, on the symbols $[n]$, unless explicitly stated otherwise. For the problem of finding a Latin square L that avoids a specific fixed array A, the intricacy is the natural number that answers the following question: "If we want to split A into avoidable arrays, what is the minimum number of arrays that we need to use?" There is some ambiguity regarding how the problem instances are partitioned in the phrasing of this question, and we will address this question below.

E-mail address: lars-daniel.ohman@math.umu.se.

0012-365X/$ – see front matter
doi:10.1016/j.disc.2012.07.003

Let $I(A)$ be the intricacy of avoiding an array A, and let $A_{m,n}$ be the class of $n \times n$ arrays with at most m entries in any cell, where $m < n$. We set $I(m, n) = \max\{I(A) : A \in A_{m,n}\}$. Evidently, for any fixed n, $I(m, n)$ grows monotonically with m. In [4] it was proven that $I(1, n) \leq 3$, and in [9] it was proven that $I(1, n) = 2$ (giving a positive resolution to Conjecture 3.3 in [4]). In what follows, we will therefore generally assume that $m > 1$.

The combinatorial construction problem of creating an $n \times n$ Latin square L that avoids an array A with at most m entries in each cell is certainly fair for $n > m$, if we say that a minimal array is an array with exactly one non-empty cell, containing at most m entries. If $m = n$, the problem is not fair, so in what follows, unless explicitly stated, we investigate $n \times n$ arrays, with at most $1 < m < n$ entries in each cell.

We will make use of the following theorems on completing partial Latin squares.

Theorem 1.3 (*Andersen and Hilton [1]*)**.** *Any partial Latin square with at most $n-1$ entries is completable. A partial Latin square with n entries is completable if and only if it does not contain any of the following configurations (up to permutations of symbols, rows and columns, and interchanging the roles of rows and columns):*

1. *(Type I) Cells $(1, 1), (1, 2), \dots, (1, m)$ contain symbols $1, \dots, m$, and cells $(2, m + 1), (3, m + 1), \dots, (n - m, m + 1)$ contain symbols $m + 1, \dots, n$.*
2. *(Type II) Cells $(1, 1), (1, 2), \dots, (1, m)$ contain symbols $1, \dots, m$, and cells $(2, m+1), (3, m+2), \dots, (n-m, n)$ all contain the symbol $m + 1$.*

We shall use the phrase "Type II" to denote a configuration as in Theorem 1.3.

Theorem 1.4 (*Ryser [13]*)**.** *Let P be an $n \times n$ partial Latin square, whose upper left $r \times s$ subsquare is completely filled, and where no other cells are filled. Then P is completable if and only if each symbol occurs at least $(r + s) - n$ times in P.*

Theorem 1.5 (*Corollary 11.4.2 from [2]*)**.** *Let P be a partial $n \times n$ Latin square, all of whose entries lie within an $r \times s$ rectangle, with $r + s \leq n$. Then P can be completed to an $n \times n$ Latin square.*

Note that for even n, Theorem 1.1 follows directly from Theorem 1.5, which is in turn a corollary of Theorem 1.4. For odd n, some additional minor work is needed, but the full Theorem 1.1 is essentially a corollary to Theorem 1.5.

Theorem 1.6 (*[10]*)**.** *Let $n = ar + t$ where $0 \leq t, r \geq a - 2 \geq 1$ and let P be a partial $n \times n$ Latin square, all of whose entries lie within a set of $a - 1$ subsquares of size $r \times r$, such that each row and column intersects at most one such subsquare. Then P can be completed to an $n \times n$ Latin square.*

Theorem 1.7 (*[10]*)**.** *Let $n = ar + t$ where $0 \leq t$, and let P be a partial $(ar + t) \times (ar + t)$ Latin square, all of whose entries lie within $\lceil \frac{a+1}{2} \rceil$ subsquares of size $r \times r$, such that each row or column intersects at most one such subsquare. Then P can be completed to an $n \times n$ Latin square.*

Finally, we shall employ a theorem by Galvin [7] on the list chromatic index of bipartite multigraphs, here rephrased in terms of constructing partial Latin squares with restrictions on which symbols may be used in which cells. We phrase a special case of the theorem in terms of avoidable arrays.

Theorem 1.8 (*Galvin [7]*)**.** *Let B be a subset of the cells of an $n \times n$ array A. Suppose that for each cell c in B, there are at most $k - 1$ other cells of B in the same row as c, and at most $k - 1$ other cells of B in the same column as c. If A has at most $n - k$ symbols in each cell, then the cells of B can be filled in such a way that no symbol is repeated in any row or column, and no cell in the completed B holds a symbol that is present in the corresponding cell of A.*

2. Single-entry and multiple-entry arrays

A *single-entry array* is an array A in which each cell holds at most one symbol ($A \in A_{1,n}$), and a *multiple-entry array* is an array in which cells may hold more than one symbol ($A \in A_{m,n}$ for some $m > 1$). There exist unavoidable single-entry arrays, for example any array containing a whole row or column of just one symbol, so the intricacy of avoiding arrays with at most one entry in each cell is not 1. If $m = n = 1$ the problem is not fair, as observed above, so we assume that $n \geq 2$ to avoid this degenerate case.

The following result is from [9], but we repeat it here for the convenience of the reader.

Theorem 2.1. *The intricacy of avoiding single-entry arrays is 2, that is $I(1, n) = 2 = \lceil \frac{n}{n-1} \rceil$.*

Proof. Let A be any $n \times n$ single-entry array on the symbols $[n]$. Initially split A into arrays B and C, where C is empty. Certainly, then, there is a Latin square L_C avoiding C. For each cell in B, move the entry to C if and only if it differs from the corresponding entry in L_C. Then L_C will still avoid C, and the entries left in B form a partial Latin square, which is completable (to L_C, for instance). Now B is avoidable, for example by L_C with symbols permuted without fixed points. $\square$

When $m \geq 2$, we must decide what we mean by a partition of an array A. Do we partition the sets of symbols in each cell, or do we merely partition the set of cells? If we allow ourselves to partition the sets of symbols in each cell, then for any $n > 1$ the intricacy of avoiding multiple-entry arrays with at most $m \leq n$ entries in any cell is 2, by the exact same argument as in the proof of Theorem 2.1.

If the partition of the array keeps entries in a cell together, which is the situation considered in the present paper, the situation is more difficult. We cannot mimic Theorem 2.1 for any $m \geq 2$, because the cells not moved to the array C may prescribe as many as $mn - n$ symbols that a given symbol must not be permuted to in the last step of the proof. We may even be so unfortunate that *no* cell can even be moved to C. If we were to proceed along the lines of this proof, it is obvious that more care would have to be taken in choosing the Latin square L_C.

3. Sparse avoidable arrays

In this section we establish the avoidability of some sparse arrays. Note that the following proposition (Proposition 3.1) is sharp, for if there are n entries, it might be the case that a whole row or a whole column is filled with one symbol, or it might be the case that one cell contains all n symbols. Such arrays are evidently unavoidable.

Proposition 3.1. *Let $A \in \mathcal{A}_{m,n}$. If the total number of entries in A, counted with multiplicities, is at most $n-1$, then A is avoidable.*

Proof. Let $a_1, a_2, \ldots, a_k$ be the non-empty cells of A, with symbol set L_i in cell a_i. We have $k \leq n-1$ and $|L_i| \leq n-k$ for all i. We may assume without loss of generality that $|L_1| \geq |L_2| \geq \cdots \geq |L_k|$. Choose a permissible symbol $b_1 \notin L_1$ for cell a_1, and forbid the use of b_1 in any cells a_j that lie in the same row or column as a_1. Do this for each i with $1 \leq i \leq k$. At step i, we must choose a symbol for cell a_i, where there are at least $n - |L_i| - (i-1)$ symbols available. Since $|L_i| \leq (n-1) - \sum_{j=1}^{i-1} |L_j|$ the number of available symbols in cell a_i is at least $n - (n-1) + \sum_{j=1}^{i-1} |L_j| - (i-1) = \sum_{j=1}^{i-1} |L_j| - i + 2 \geq 1$. When we are finished, the chosen b_i constitute a partial Latin square with at most $n-1$ entries, that is completable by Theorem 1.3. The completed Latin square certainly avoids A. $\square$

If we prescribe that no cell of A may contain more than $n-1$ symbols, we may prove the following variant of the above proposition.

Proposition 3.2. *Let $1 \leq m \leq n-1$, and let $A \in \mathcal{A}_{m,n}$. If the total number of non-empty cells in A is at most $n-m$ then A is avoidable.*

Proof. If we can avoid the non-empty cells then Theorem 1.3 ensures that the partial array found in this way can be completed to a full Latin square. To avoid the non-empty cells, we choose a permitted symbol for each non-empty cell, in arbitrary order, and forbid its use in any other non-empty cell in the same row or column. Each non-empty cell allows at least $n-m$ different symbols, and there are $n-m$ of them, so at step i there will be at least $(n-m)-(i-1) \geq 1$ available symbols, and the process will not break down. $\square$

4. Lower bounds on $I(m, n)$ when $m \geq 2$

To bound the intricacy from below, we need to find arrays that are unavoidable unless partitioned into "many" parts.

Proposition 4.1. *If $m > \frac{n}{2}$ then $I(m, n) \geq 3$.*

Proposition 4.2. *If $m = n-1$, then $I(m, n) \geq n$.*

One type of example that can be used to prove both propositions is an $n \times n$ array A with entries $1, 2, \ldots, m$ in each cell in the first column, as follows:

For the first of these two propositions, if $m > \frac{n}{2}$ and we partition the cells of the first column into two parts, one of the parts will get at least $\frac{n}{2}$ of the cells, say k cells, effectively blocking the use of symbols $1, \ldots, m$ in at least k cells in the first column, so these symbols can only be used in at most $n-k < m$ cells, which is impossible. Therefore, the intricacy is not 2, and hence at least 3.

For the second proposition, if we partition A into $n-1$ parts, the pigeonhole principle gives that at least one of the parts, say B, contains at least two of the cells from the first column. Again, this means that $m = n-1$ different symbols must be used in the at most $n-2$ free cells in the first column of B, which is obviously a contradiction.

Using the pigeonhole principle in this way, we get a whole range of intermediate results of these two propositions.

Theorem 4.3. *For $1 \leq m < n$, it holds that $I(m, n) \geq \left\lceil \frac{n}{n-m} \right\rceil$.*

Note that, in particular, for $m \leq n/2$ we have $I(m, n) \geq 2$.

5. Upper bounds on $I(m, n)$ for $m \geq 2$

To bound the intricacy $I(m, n)$ from above we must, given an arbitrary array $A \in \mathcal{A}_{m,n}$, describe how to partition it into "few" avoidable arrays. One useful way of partitioning arrays is by means of generalized diagonals.

A *generalized diagonal* in an $n \times n$ array is a set of n cells, such that no two cells share a row or column. In what follows, we will simply write "diagonal" for "generalized diagonal". Any array with at most $n-2$ entries in each cell may be decomposed into n diagonals such that on these diagonals, the use of at most $n-2$ symbols is forbidden in each cell. Obviously, for each such part we can construct partial Latin squares (with non-empty cells on the diagonal) that avoid the $n-2$ symbols in each of the relevant cells, and that do not use one single symbol exactly $n-1$ times. By Theorem 1.3, then, each such array is completable, and thus for $m \leq n-2$ we have $I(m, n) \leq n$.

From now on, we will call a diagonal with exactly $n-1$ identical sets of $n-1$ symbols, where the last cell contains the symbol missing from the $n-1$ other cells, a *bad* diagonal. If we could decompose any $n \times n$ array A with at most $n-1$ symbols in each cell into diagonals, none of which is bad, we would have established that $I(n-1, n) \leq n$, by Theorem 1.3. However, it is easy to find examples of arrays where this is not possible, and therefore we must in general modify the partition in order to make each part avoidable, as in the proof of the following theorem.

Theorem 5.1. *For $n \geq 4$, it holds that $I(n-1, n) = n$.*

Proof. By Theorem 4.3, $I(n-1, n) \geq n$, so it remains to establish that $I(n-1, n) \leq n$.

We partition A into the n diagonals parallel to the diagonal consisting of entries (i, i) (the main diagonal). If there are no bad diagonals, we are finished. If not, let the bad diagonals be $D_1, \ldots, D_\ell$. Let s_i be the single symbol that is not forbidden in $n-1$ cells of D_i, and $\sigma_i \neq s_i$ be a symbol not forbidden in the last cell, $c_i \in D_i$. We will first prove that if there exist two bad diagonals, then we can modify them both to make them avoidable.

If there is a pair of bad diagonals D_i and D_j with $s_i = s_j$ (and possibly $\sigma_i = \sigma_j$), we can set $D_i' = (D_i \setminus \{c_i\}) \cup \{c_j\}$ and $D_j' = (D_j \setminus \{c_j\}) \cup \{c_i\}$. After this move, for each modified diagonal D_i' and D_j', the partial Latin square defined by the single allowed symbols in the non-empty cells is completable, by Theorem 1.3. We may thus assume that all s_i are distinct.

If there is a pair of bad diagonals D_i and D_j such that $\sigma_i = s_j$ (and possibly $\sigma_j = s_i$ as well), we may proceed as follows. Let $c_{j_1}, c_{j_2} \in D_j$ be the two non-empty cells that lie in the same row or column as c_i. Note that one of the two cells c_{j_1}, c_{j_2} may happen to be c_j. We set $D_i' = (D_i \setminus \{c_i\}) \cup (\{c_{j_1}, c_{j_2}\} \setminus \{c_j\})$ and $D_j' = (D_j \cup \{c_i\}) \setminus (\{c_{j_1}, c_{j_2}\} \setminus \{c_j\})$.

Now, the partial Latin square defined by the single allowed symbols in the $n-1$ or n non-empty cells of D_j' is completable by Theorem 1.3, so D_j' is avoidable. The single allowed symbols in the n or $n+1$ non-empty cells of D_i' again define a partial Latin square, which can be completed. To see this, observe that by setting the symbol s_i in every cell of the original D_i, and setting the symbol s_j in every cell moved to D_i' from D_j, we form a partial Latin square that respects the conditions given by the non-empty cells of D_i'. This partial Latin square can easily be completed. We may thus assume that all the s_i are distinct, and that $s_i \neq \sigma_j$ for all bad diagonals D_i and D_j.

Finally, if there is a pair of bad diagonals D_i and D_j such that $\sigma_i \neq s_j$ and $s_i \neq \sigma_j$, we may set $D_i' = (D_i \setminus \{c_i\}) \cup \{c_j\}$ and $D_j' = (D_j \setminus \{c_j\}) \cup \{c_i\}$. It is easy to check that both D_i' and D_j' are avoidable, by Theorem 1.3.

We may therefore suppose that there is exactly one bad diagonal D_1 remaining, which we may take to be the main diagonal without loss of generality. To complete the proof, we must reform D_1 so as to make it avoidable. Note that the other parts of the partition of A may no longer be diagonals parallel to the main diagonal, since some parts may have started out as bad diagonals and been reformed. In particular, there may be some parts containing $n+1$ cells.

Take some arbitrary other part of the partition, say D, with $|D| \leq n$. We observe that if there is no cell $c_0 \in D$ where a symbol $s_0 \neq s_1$ is allowed, both $D_1 \setminus \{c_1\}$ and $D \cup \{c_1\}$ will be avoidable, and we are finished.

Now consider the cells in D in the same row or column as c_1. At most two of these cells allow only the symbol σ_1, since D is avoidable. Suppose there are two such cells, say c_0' and c_0''. Then both $(D_1 \setminus \{c_1\}) \cup \{c_0', c_0''\}$ (with only two distinct allowed symbols) and $(D \setminus \{c_0', c_0''\}) \cup \{c_1\}$ (with at most $n-1$ non-empty cells) will be avoidable. We may therefore suppose that there is at most one such cell, c_0'.

Now, take a cell $c_0 \in D$ where some symbol $s_0 \neq s_1$ is allowed, and set $D' = (D \setminus \{c_0\}) \cup \{c_1\}$, and $D_1' = (D_1 \setminus \{c_1\}) \cup \{c_0\}$, so that we have $|D'| \leq n$ and $|D_1'| = n$. Then D_1' will be avoidable by Theorem 1.3.

If D' is avoidable, we are finished. If D' is unavoidable, it is either

(A) because there is a cell $c_0' \in D'$ in the same row or column as c_1 permitting only the symbol σ_1, or
(B) because $|D_1'| = n$ and the single symbols not forbidden in the non-empty cells of D' define a Type II configuration involving, say, symbols σ_a and σ_b (see Fig. 1).

If (A) is the case, we set $D'' = D' \setminus \{c_0'\}$ and $D_1'' = D_1' \cup \{c_0'\} = (D_1 \setminus \{c_1\}) \cup \{c_0, c_0'\}$. Then D'' will be avoidable, since $|D''| < n$. Also, D_1'' will be avoidable. To see this, note that the non-empty cells in D_1'' specify the use of the symbol s_1 in the cells of D_1, and the entries in two additional cells, $c_0', c_0 \notin D_1$. Suppose the entry in c_0 is s_a and the entry in c_0' is s_b. To complete the partial Latin square thus specified, it suffices to find two disjoint diagonals D_a and D_b through cells c_0 and c_0', respectively, that do not intersect D_1, for we can then set the symbol s_a in all the cells of D_a and symbol s_b in all the cells of D_b, and then easily complete the partial Latin square thus specified. Since $n \geq 4$, such a pair of diagonals can easily be found, and we are finished.

If (B) is the case, we have the situation presented in Fig. 1, for some suitable σ_a and σ_b. In this case, we observe that either $s_1 \neq \sigma_a$ or $s_1 \neq \sigma_b$ (or both). In the case $s_1 \neq \sigma_a$, we let c_a be the cell containing σ_a, and set $D'' = D' \setminus \{c_a\}$ and $D_1'' = D_1' \cup \{c_a\}$. Then D'' is avoidable, because $|D''| < n$.

To see that D_1'' is also avoidable, observe that D_1'' consists of $n-1$ identical entries on a diagonal, permitting only the use of the symbol s_1, and two additional cells, c_0 and c_a off this diagonal, permitting only the use of the symbol s_0 and σ_a,

σ_1	σ_a			
		σ_b		
			σ_b	
				$\ldots$

Fig. 1. Case (B), a Type II configuration in D'. Note that the cell c_1 holds the symbol σ_a, and that $\sigma_1 \neq \sigma_a$.

respectively. Since $n \geq 4$, we can easily find two pairwise disjoint diagonals D_0 and D_a, that are also disjoint from the original D_1, having the property that $c_0 \in D_0$, and $c_a \in D_a$. We observe that the partial Latin square having symbol s_1 on the original diagonal D_1, s_0 on D_0 and σ_a on D_a respects the restrictions from D_1'', and is completable.

For the case $s_1 \neq \sigma_b$, we may argue like for the case where $s_1 \neq \sigma_a$: Let c_b be one of the cells in D' where σ_b is permitted, and set $D'' = D' \setminus \{c_b\}$ and $D_1'' = D_1' \cup \{c_b\}$. Then both D'' and D_1'' will be avoidable, and we are finished. □

By Theorem 2.1, $I(1, 2) = 2$, so the only remaining case is $n = 3$. It is fairly straightforward to check by exhaustive computer search that $I(2, 3) = 3$, so in fact $I(n-1, n) = n$ for all $n \geq 2$. Note that since $I(m, n)$ for fixed n increases monotonically with m, it follows immediately from Theorem 5.1 that $I(m, n) \leq n$ for all $m < n$. If m is small in comparison to n, more effective upper bounds on $I(m, n)$ can be found.

Theorem 5.2. *Let $m^2 - m \leq (n-1)(n+2)/2n$ and $n \geq 4$. Then $I(m, n) \leq 3 = \left\lceil \frac{n}{n-m} \right\rceil + 1$.*

Proof. Let A be the array that is to be avoided, and let L and L' be two Latin squares such that in each cell, L and L') have different entries. For any permutation p of the symbols in L and L', we then also have that the corresponding permuted Latin squares, $p(L)$ and $p(L')$, differ in each cell.

By a slight abuse of notation, let $p(L) \cap A$ be the set of cells where the entry in $p(L)$ is present in A, and $(p(L) \cap A) \cup (p(L') \cap A)$ be the set of cells where either the entry in $p(L)$ or the entry in $p(L')$ is present in A.

By a standard double count, we then have the following, where Π_n denotes the set of permutations of $[n]$, and (i, j) is a set of coordinates:

$$\begin{aligned}\sum_{p \in \Pi_n} |(p(L) \cap A) \cup (p(L') \cap A)| &= \sum_{(i,j) \in A} |\{p \in \Pi_n : (i, j) \in (p(L) \cap A) \cup (p(L') \cap A)\}| \\ &= n^2 m(m-1)(n-2)!\end{aligned}$$

Taking averages over Π_n, noting that $|\Pi_n| = n!$, we see that there exists a permutation p_0 with the property that $|(p_0(L) \cap A) \cup (p_0(L') \cap A)| \leq (m^2 - m)n/(n-1)$. In other words, there are at most $(m^2 - m)n/(n-1)$ cells in A that are not avoided by either $p_0(L)$ or $p_0(L')$.

Now, observe that for $m \geq 3$ it holds that if $(m^2 - m)n/(n-1) \leq \frac{n}{2} + 1$, then certainly $m \leq \frac{n}{2} - 1$. The $(m^2 - m)n/(n-1)$ cells are therefore avoidable by Proposition 3.2. If $m = 2$ there are at most $(2^2 - 2)n/(n-1) < 3$ cells not avoided by $p_0(L)$ or $p_0(L')$, each containing at most two symbols. Since $n \geq 4$ this is obviously avoidable. For $m = 1$, we have $I(1, n) = 2 < 3$ by Theorem 2.1. □

Solving for m in the inequality, we get $m \leq \frac{1}{2} + \sqrt{\frac{n}{2} + \frac{3}{4} - \frac{1}{n}}$, so if m and n satisfy this condition, we have, for $n \geq 4$, that $I(m, n) \leq 3$. Note that applying permutations to the rows and columns of L and L' does not improve these calculations. Also, using three or more pairwise disjoint Latin squares is ineffective in comparison with other methods. For example, Theorems 1.5 and 1.6 can be applied to prove the following three theorems.

Theorem 5.3. *Let $m \leq \frac{n}{2}$. Then $I(m, n) \leq 4 = \left\lceil \frac{n}{n-m} \right\rceil + 2$.*

Proof. If n is even, split the array of forbidden symbols into the four quadrants. For each of these quadrants, all non-empty cells can be avoided by Theorem 1.8. The partial Latin squares found in this way are completable, by Theorem 1.5.

If n is odd, split the array into four non-overlapping $\frac{n-1}{2} \times \frac{n+1}{2}$ or $\frac{n+1}{2} \times \frac{n-1}{2}$ rectangles, located in the corners of the array to be avoided, such that only the cell $c = \left(\frac{n+1}{2}, \frac{n+1}{2}\right)$, in the middle of the array, is not covered. This cell is the fifth part. The non-empty cells in all five parts are avoidable, by Theorem 1.8, and the partial Latin squares found in this way are completable, by Theorem 1.5.

In fact, the cell c may be added to any one of the other four parts, say part P. Theorem 1.8 again ensures that this new part may be avoided, and the resulting partial Latin square found in this way is completable, which can be seen by taking the Latin square L that completes the partial Latin square L_P that avoids P, and permute those rows and columns of L that do not intersect L_P to match the symbols permitted in c. □

Theorem 5.4. *Let $m \leq n + 1 - \sqrt{n+1}$. If $\frac{n}{n-m}$ is an integer, then $I(m, n) \leq \frac{n}{n-m} + 2$. Otherwise $I(m, n) \leq \left\lceil \frac{n}{n-m} \right\rceil + 4$.*

Proof. Let $a = \lceil \frac{n}{n-m} \rceil$. Then $a \geq 2$, since $m > 0$, and $m \leq n + 1 - \sqrt{n+1}$ ensures that $n - m \geq a - 2$.

First, suppose that $\frac{n}{n-m}$ is an integer, and let A be the $n \times n$ array to be avoided. We split A into a^2 square sectors, with side lengths $n - m$. In the $a \times a$ grid formed in this way, consider the a diagonals parallel to the main diagonal.

If we can remove at least one subsquare from each of the a diagonals, the non-empty cells in each such diagonal can be avoided, by Theorem 1.8, and the partial Latin squares found in this way can be completed by Theorem 1.6, since $n - m \geq a - 2$.

Removing at least one subsquare from each diagonal is easily done: Remove the two sets of subsquares $P_1 = \{(1, a), (2, a-1), \ldots, (a-1, 2)\}$ and $P_2 = \{(1, a-1), (2, a-2), \ldots, (a-1, 1)\}$, and let P_1 and P_2 be new parts in the partition. Here (i, j) signifies the subsquare in the $a \times a$ grid with coordinates (i, j). We will then have partitioned A into $a + 2$ parts. What remains is to argue that P_1 and P_2 are avoidable. Since the non-empty cells in both P_1 and P_2 are avoidable by Theorem 1.8, and both P_1 and P_2 satisfy the conditions in Theorem 1.6, both P_1 and P_2 are avoidable. In this case, therefore, $I(m, n) \leq \frac{n}{n-m} + 2$.

If $\frac{n}{n-m}$ is not an integer, we split A into $(a-1)^2$ square sectors, with side lengths $n - m$, starting in the upper left hand corner, and smaller rectangles $\left(\text{size } (n-m) \times t \text{ or } t \times (n-m) \text{ for } t = n - (n-m)\left\lfloor \frac{n}{n-m} \right\rfloor\right)$ and one smaller square (size $t \times t$) in the lower right corner.

We now consider the a diagonals in the $a \times a$ grid that are parallel to the main diagonal. Note that each such diagonal will then contain either at least one rectangle, or the $t \times t$ subsquare. If we can remove at least two subrectangles (where we take this term to include subsquares) from each of the a diagonals, the non-empty cells in each such reduced diagonal can be avoided, by Theorem 1.8, and the partial Latin squares found in this way can be completed by Theorem 1.6, since $n - m \geq a - 2$.

Removing at least two subrectangles from each diagonal can be done for $a \geq 4$ by setting $P_1 = \{(1, a-1), (2, a-2), \ldots, (a-2, 2)\}$, $P_2 = \{(1, a-2), (2, a-3), \ldots, (a-2, 1)\}$, $P_3 = \{(a, 1), (a-1, 2), \ldots, (3, a-2)\}$ and $P_4 = \{(a, 2), (a-1, 3), \ldots, (3, a-1)\}$, and letting these sets form parts in the partition of A. Each P_i is avoidable, by Theorems 1.8 and 1.6 as above.

If $a = 3$, the P_i as specified above only remove one subrectangle from each diagonal, so instead we set $P_1 = \{(1, 2), (2, 1)\}$, $P_2 = \{(1, 1), (2, 2)\}$, $P_3 = \{(3, 1), (4, 2)\}$ and $P_4 = \{(3, 2), (4, 1)\}$, thus removing two subsquares from each diagonal. Again, the P_i are avoidable by Theorems 1.8 and 1.6.

If $a = 2$, then $m \leq \frac{n}{2}$, so by Theorem 5.3, $I(n, m) \leq \lceil \frac{n}{n-m} \rceil + 2$. Thus in the case where $\frac{n}{n-m}$ is not an integer, $I(m, n) \leq \left\lceil \frac{n}{n-m} \right\rceil + 4$. □

To close the gap between $(n+1) - \sqrt{n+1}$ and $n - 2$, we prove the following.

Theorem 5.5. *Let $m \leq n - 2$. Then $I(m, n) \leq 2\left\lceil \frac{n}{n-m} \right\rceil$.*

Proof. Write n as $n = a(n-m) + t$ with $t < a$. Split the array A into an $(a+1) \times (a+1)$ grid of squares of size $(n-m) \times (n-m)$, with $(n-m) \times t$ and $t \times (n-m)$ rectangles along the right and bottom edges, and a $t \times t$ square in the lower right corner. Separate A into avoidable parts, each consisting of half a diagonal in the grid. The non-empty cells in each of these parts are avoidable by Theorem 1.8, and the partial Latin squares thus found are completable, by Theorem 1.7. □

6. Concluding remarks

To sum up, we have the following table, where $\left\lceil \frac{n}{n-m} \right\rceil \leq I(m, n) \leq n$.

m	$I(m, n)$
$m = 1$	$I(m, n) = \left\lceil \frac{n}{n-m} \right\rceil$
$m \leq \frac{1}{2} + \sqrt{n/2 + 3/4 - 1/n}$	$I(m, n) \leq \left\lceil \frac{n}{n-m} \right\rceil + 1$
$m \leq n/2$	$I(m, n) \leq \left\lceil \frac{n}{n-m} \right\rceil + 2$
$m \leq n + 1 - \sqrt{n+1}$, $\frac{n}{n-m} \in \mathbb{N}$	$I(m, n) \leq \left\lceil \frac{n}{n-m} \right\rceil + 2$
$m \leq n + 1 - \sqrt{n+1}$	$I(m, n) \leq \left\lceil \frac{n}{n-m} \right\rceil + 4$
$m \leq n - 2$	$I(m, n) \leq \left\lceil \frac{n}{n-m} \right\rceil \cdot 2$
$m = n - 1$	$I(m, n) = \left\lceil \frac{n}{n-m} \right\rceil$

On the basis of this evidence, it seems reasonable to propose the following conjecture.

Conjecture 6.1. $I(m, n) = \left\lceil \frac{n}{n-m} \right\rceil$ *for all* $1 \leq m \leq n - 1$.

The specific instance $I(2, 4) = 2$ should be tractable by computer, but would require some ingenuity in reducing the number of instances.

In a sense, the condition that no cell hold more than m forbidden symbols is a very asymmetric condition, in view of the row–column–symbol symmetry in Latin squares. It might therefore be considered more natural to also prescribe that no symbol be forbidden in more than m cells in any row or in any column. This problem was suggested by Häggkvist in [8]. Let $A_{m,m,m,n}$ be the class of $n \times n$ arrays where no cell holds more than m symbols, and no symbols appears more than m times in any row or column. Häggkvist conjectured, in these terms, that if $m < n/3$, then any $A \in A_{m,m,m,n}$ is avoidable. For $m = 1$, this problem has been studied in [4] and resolved in the affirmative in [3,11] for $n \geq 4$. For general m, it has been studied in [5].

Acknowledgments

The author would like to thank Martin Berglund from the Department of Computing Science, Umeå University, for assistance with computationally establishing that $I(2, 3) = 3$, and an anonymous referee for many helpful comments.

References

[1] L.D. Andersen, A.J.W. Hilton, Thank evans!, Proc. Lond. Math. Soc. (3) 47 (3) (1983) 507–522.
[2] A. Asratian, T. Denley, R. Häggkvist, Bipartite Graphs and Their Applications, Cambridge University Press, 1998.
[3] N.J. Cavenagh, Avoidable partial Latin squares of order $4m + 1$, Ars Combin. 95 (2010) 257–275.
[4] A.G. Chetwynd, S.J. Rhodes, Avoiding partial Latin squares and intricacy, Discrete Math. 177 (1997) 17–32.
[5] J. Cutler, L–D. Öhman, Latin squares with forbidden entries, Electron. J. Combin. 13 (1) (2006) R47.
[6] D.E. Daykin, R. Häggkvist, Problem 6347, Amer. Math. Monthly 88 (1981) 446.
[7] F. Galvin, The list chromatic index of a bipartite multigraph, J. Combin. Theory Ser. B 63 (1995) 153–158.
[8] R. Häggkvist, A note on Latin squares with restricted support, Discrete Math. 75 (1989) 253–254.
[9] L–D. Öhman, The intricacy of avoiding arrays is 2, Discrete Math. 306 (2006) 531–532.
[10] L–D. Öhman, A note on completing Latin squares, Australas. J. Combin. 45 (2009) 117–124.
[11] L–D. Öhman, Partial Latin squares are avoidable, Ann. Comb. 15 (3) (2011) 485–497.
[12] W.E. Opencomb, On the intricacy of combinatorial construction problems, Discrete Math. 50 (1984) 71–97.
[13] H.J. Ryser, A combinatorial theorem with an application to Latin rectangles, Proc. Amer. Math. Soc. 2 (1951) 550–552.